Energetic Consistency and Symmetry in Gyrokinetic Field Theory

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Outline

- **Gyrokinetic Field Theory**
  — system Lagrangian in canonical form
  — how to use functional derivatives
  — symmetry in Lagrangian: time/energy, axisymmetry/momentum

- **Momentum Conservation in Gyrokinetic Theory**
  — canonical versus plasma momentum
  — phase space continuity, transport equations
  — role of charge conservation $\leftrightarrow$ FS-averaged radial current

- **Momentum Transport**
  — correspondence to the MHD limit
  — symmetric structure, dynamics (not spatial)
  — contributions of all orders have same symmetry properties

Gyrokinetic Theory and Momentum

- field theory for fluids with special emphasis on momentum
  - route to consistency against ExB energy (Pfirsch and Correa-Restrepo 1996-1998)

- since 2000:
  - field theory (Sugama, Brizard 2000)
  - full capture of MHD including equilibrium (Qin 2000)
  - major review: Brizard and Hahm, Rev Mod Phys (2007)
  - dynamical "equilibrium" flows (Miyato and Scott 2009)
  - full route to Maxwell equations (Pfirsch, Correa-Restrepo, Madsen 2005-10)

- momentum conservation and transport
  - role of energetic consistency, vorticity conservation/transport (Scott 2010)
  - role of dimensional reduction in momentum conservation (Brizard/Tromko 2011)

- any serious GK theory is a field theory with support by symmetry principles
  - conservation from any Lagrangian in canonical form is rigorous
  - proof of toroidal momentum transport involves charge conservation
  - gyrocenter charge is the same thing as generalised vorticity

Basic Structure of GK Lagrangian

- symplectic part and Hamiltonian for particles, plus magnetic field energy
  ◦ quantities $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{W}$ are magnetic geometry (here, arbitrary)

$$L = \sum_{\text{sp}} \int d\Lambda\, f \left[\left(\frac{e}{c}\mathbf{A} + p_z \mathbf{b} - \frac{mc}{e}\mu\mathbf{W} \right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - H \right] - \int d\mathcal{V}\, \frac{B_\perp^2}{8\pi}$$

- Hamiltonian depends on phase space coords and time/space-dependent fields

$$H = H(\mathbf{R}, p_z, \mu, \phi, A_\parallel) \qquad \{\phi, A_\parallel\} = \{\phi, A_\parallel\}(t, \mathbf{R}) \qquad \frac{\partial}{\partial\vartheta}(\text{anything}) = 0$$

- all field dependence is Lie-transformed into H (strictly)
  ◦ no $\partial/\partial t$ on fields in gyrokinetic (GK) eqn, no gyrophase (ϑ) dependence

- particle motion: Euler-Lagrange (E-L) eqs for gyrocenter coordinates

- GK eqn: Liouville theorem $\rightarrow$ distribution function f

- polarisation/induction eqs: E-L eqs for field potentials

Importance of Symmetry of GK Lagrangian

- symplectic part and Hamiltonian for particles, plus magnetic field energy

$$L = \sum_{\rm sp} \int d\Lambda \, f \left[\left(\frac{e}{c}\mathbf{A} + p_z \mathbf{b} - \frac{mc}{e}\mu\mathbf{W} \right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - H \right] - \int d\mathcal{V} \, \frac{B_\perp^2}{8\pi}$$

- all dependent variable involvement is in time component only
- symplectic part is strictly static and axisymmetric (contains geometry information)
- conservation laws follow: time symmetry $\rightarrow$ energy conservation
- toroidal angle (axi-)symmetry $\rightarrow$ toroidal momentum conservation
- all approximations appear in the derivation of the Lagrangian (Lie transforms)
- approximations are subect to exact preservation of the symmetry
- if this symmetry setup is rigorous, the results are rigorous

Lagrangian Density

- we can define a Lagrangian density $\mathcal{L}$ as follows

$$L \equiv \int d\mathcal{V}\, \mathcal{L}$$

- hence with the particles we have the species sum and velocity integration

$$\mathcal{L} = \mathcal{L}_f + \sum_{\mathrm{sp}} \int d\mathcal{W}\, f\, L_p$$

- here, $\mathcal{L}_f$ is the *field Lagrangian density* and L_p is the *particle Lagrangian*

$$L_p = \left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b} - \frac{mc}{e}\mu\mathbf{W}\right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - H$$

$$\mathcal{L}_f = -\frac{B_\perp^2}{8\pi} \qquad \text{an arbitrary function of } A_\parallel \text{ through gradients}$$

- electric energy enters H only (ExB kinetic), not $\mathcal{L}_f$ (electric field)
  - this is *quasineutrality* (basic assumption: ExB kinetic energy $\gg$ field energy)

What is Canonical Form?

- **Canonical Form:** all dependence on fields$(\mathbf{x}, t)$ is in time component

- time part of particle Lagrangian, for example with $\phi = \phi(\mathbf{x}, t)$ (and neglecting $\mathbf{W}$)

$$L_p = \left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b}\right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - \left(eJ_0\phi - \frac{mc^2}{2B^2}\left|\nabla_\perp\phi\right|^2\right)$$

or also in field part of total Lagrangian, for example with (linearised polarisation)

$$L = \sum_{\mathrm{sp}} \int d\Lambda\, f\left[\left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b}\right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - eJ_0\phi\right] + \int d\mathcal{V}\, n_0\frac{mc^2}{2B^2}\left|\nabla_\perp\phi\right|^2$$

- with $\phi = \phi(\mathbf{R}, t)$ the following is *not* in canonical form

$$L_p = \left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b} + \frac{c}{B}\mathbf{b}\times\nabla\phi\right) \cdot \dot{\mathbf{R}} + \frac{mc}{e}\mu\dot{\vartheta} - \left(eJ_0\phi + \frac{mc^2}{2B^2}\left|\nabla_\perp\phi\right|^2\right)$$

Why is Canonical Form Important?

- geometry: axisymmerty $\rightarrow$ simple form of momentum evolution
- in explicit $RZ\phi$-coordinates

$$L_p = \cdots + \left(\frac{e}{c}\mathbf{A} + p_z\mathbf{b}\right) \cdot \dot{\mathbf{R}} \cdots - H$$

becomes

$$L_p = \cdots + \left(\frac{e}{c}A_\varphi + p_z b_\varphi\right) \cdot \dot{\varphi} - H$$

- Euler-Lagrange equation for φ involves only these two pieces
  - ∘ particle Lagrangian is axisymmetric except for $H = H(\ldots, \varphi)$
  - ∘ the only other place φ occurs is as $\dot{\varphi}$, part of $\dot{\mathbf{R}}$
  - ∘ hence we have $(d/dt)(\partial L/\partial\dot{\varphi}) = \partial L/\partial\varphi$, or

$$\frac{dP_\varphi}{dt} = -\frac{\partial H}{\partial\varphi} \qquad \text{where} \quad P_\varphi = \frac{e}{c}A_\varphi + p_z b_\varphi$$

- this equation stands behind all momentum conservation proofs
  - ∘ it holds **if and only if** L is in canonical form

Why is Canonical Form Relevant?

- gyrokinetics is a **gauge transform** of the **particle-field system**
  ◦ not a story about averages and orbits

- coordinate transform $\mathbf{x} \to \mathbf{R} + \mathbf{a}$ with choice of $\mathbf{a}$, and addition of dS to $L\,dt$

$$L_p = \left(\frac{e}{c}\mathbf{A} + mU\mathbf{b} + m\mathbf{w} + m\frac{c}{B}\mathbf{b}\times\nabla\phi\right)\cdot\dot{\mathbf{x}} - e\phi - \frac{1}{2m}\left(mU\mathbf{b} + m\mathbf{w} + m\frac{c}{B}\mathbf{b}\times\nabla\phi\right)^2$$

with choice of

$$\mathbf{a} = -\frac{1}{\Omega}\mathbf{b}\times\left(\mathbf{w} + \frac{c}{B}\mathbf{b}\times\nabla\phi\right) \qquad\qquad \Omega = eB/mc$$

and some choices of $d(\,)/dt$, becomes

$$L_p = \left(\frac{e}{c}\mathbf{A} + mU\mathbf{b}\right)\cdot\dot{\mathbf{R}} + \frac{mc}{e}\mu\left(\dot{\vartheta} - \mathbf{W}\cdot\dot{\mathbf{R}}\right) - \left(m\frac{U^2}{2} + \mu B + e\phi - \frac{mc^2}{2B^2}\left|\nabla_\perp\phi\right|^2 + \mathrm{FLR}\right)$$

where $\mathbf{W}$ is a geometric piece preserving gyro-gauge invariance under $\vartheta \to \vartheta + \alpha(\mathbf{R})$

canonical form can always be recovered via gauge transform

Canonical Toroidal Momentum Distribution

- distribution function f satisfies Liouville Theorem

$$\dot{f} = 0 \implies \frac{\partial f}{\partial t} + \dot{\mathbf{Z}}_p \cdot \frac{\partial f}{\partial \mathbf{Z}_p} = 0$$

- incompressible form with phase space volume element

$$\frac{\partial}{\partial t}(f\sqrt{g}B_\parallel^*) + \frac{\partial}{\partial \mathbf{Z}_p} \cdot \left(f\sqrt{g}B_\parallel^* \dot{\mathbf{Z}}_p\right) = 0$$

- using this, multiply equation for P_φ by $fB_\parallel^*$ to find (using $\dot{P}_\varphi = -\partial H/\partial\varphi$)

$$\frac{\partial}{\partial t}(f\sqrt{g}B_\parallel^* P_\varphi) + \frac{\partial}{\partial \mathbf{Z}_p} \cdot \left(f\sqrt{g}B_\parallel^* P_\varphi \dot{\mathbf{Z}}_p\right) = -(f\sqrt{g}B_\parallel^*)\frac{\partial H}{\partial \varphi}$$

- this is the simple derivation of toroidal momentum phase space continuity

same result using brackets

- express gyrokinetic equation as a bracket (algebraically linear in both H and f)

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \{H, f\} = 0$$

- multiply by P_φ and use bracket algebra

$$\frac{\partial}{\partial t} f P_\varphi + \{H, f P_\varphi\} = f\{H, P_\varphi\} = f\frac{dP_\varphi}{dt} = -f\frac{\partial H}{\partial \varphi}$$

- same as advection form, also write divergence form

$$\frac{\partial}{\partial t} f P_\varphi + \dot{\mathbf{Z}}_p \cdot \frac{\partial}{\partial \mathbf{Z}_p} f P_\varphi = -f\frac{\partial H}{\partial \varphi}$$

$$\frac{\partial}{\partial t} f P_\varphi + \frac{1}{\sqrt{g} B_\parallel^*} \frac{\partial}{\partial \mathbf{Z}_p} \cdot \left(\sqrt{g} B_\parallel^* \dot{\mathbf{Z}}_p \, f P_\varphi \right) = -f\frac{\partial H}{\partial \varphi}$$

A Result using Functional Derivatives

- assume

$$L = \sum_{\rm sp} \int d\Lambda\, f(\mathbf{P}\cdot\dot{\mathbf{Z}}_p - H) + \int d\mathcal{V}\,\mathcal{L}_f \qquad\qquad H = H(A, \nabla A, \nabla_\perp^2 A)$$

- do chain rule on derivatives using these dependences for each of several fields A
- use field equation $\delta L/\delta A = 0$ to eliminate most terms, find

$$\sum_{\rm sp}\int d\mathcal{W}\, f\frac{\partial H}{\partial\varphi} = \frac{\partial\mathcal{L}_f}{\partial\varphi} - \frac{\partial}{\partial\varphi}\nabla\cdot\frac{\partial\mathcal{L}}{\partial\nabla_\perp^2 A}\nabla_\perp A$$

$$+\nabla\cdot\left[-\frac{\partial\mathcal{L}}{\partial\nabla A}\frac{\partial A}{\partial\varphi} + \left(\frac{\partial}{\partial\varphi}\frac{\partial\mathcal{L}}{\partial\nabla_\perp^2 A}\right)\nabla_\perp A + \frac{\partial A}{\partial\varphi}\left(\nabla_\perp\frac{\partial\mathcal{L}}{\partial\nabla_\perp^2 A}\right)\right]$$

- these are all summed over the field variables A (usually, ϕ and $A_\|$)
- for momentum transport under flux surface average,
  the last line gives all the wave-wave nonlinear fluxes

Canonical Toroidal Momentum Conservation

- simplified version assumes field dependence like $H(\phi, A_\|, \nabla\phi, \nabla A_\|)$
  - (general: add $\nabla_\perp^2$ dependence, adds to divergence term)

- apply velocity space integral to annihilate $\partial/\partial p_z$ and $\partial/\partial\mu$
  - note that $\int d\mathcal{W}/B_\|^*$ commutes past ∇ and that $(\partial/\partial\mathbf{Z}_p)\cdot(\sqrt{g}\mathbf{S}) = \sqrt{g}\nabla\cdot\mathbf{S}$

$$\frac{\partial}{\partial t}\sum_{\rm sp}\int d\mathcal{W}\, f P_\varphi + \nabla\cdot\sum_{\rm sp}\int d\mathcal{W}\, f P_\varphi \dot{\mathbf{R}} = -\sum_{\rm sp}\int d\mathcal{W}\, f\frac{\partial H}{\partial\varphi}$$

- use the result on functional derivatives to evaluate $\partial H/\partial\varphi$ term

$$\sum_{\rm sp}\int d\mathcal{W}\, f\frac{\partial H}{\partial\varphi} = \frac{\partial\mathcal{L}_f}{\partial\varphi} - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\phi}\frac{\partial\phi}{\partial\varphi} + \frac{\partial\mathcal{L}}{\partial\nabla A_\|}\frac{\partial A_\|}{\partial\varphi}\right)$$

- integration over space annihilates $\partial/\partial\varphi$ and $\nabla\cdot$, yielding **momentum conservation**

recall $\quad d\mathcal{V}\otimes d\mathcal{W} = d\Lambda \qquad \Longrightarrow \qquad \frac{\partial}{\partial t}\sum_{\rm sp}\int d\Lambda\, f P_\varphi = 0$

Energy Conservation

- simplified version assumes field dependence like $H(\phi, A_\parallel, \nabla\phi, \nabla A_\parallel)$
  - (general: add $\nabla_\perp^2$ dependence, adds to divergence term)

- same calculation as before, but with H and $\partial/\partial t$ instead of P_φ and $\partial/\partial\varphi$
  - some bracket manipulation to get $fH\dot{\mathbf{R}}$ together under the divergence

$$\frac{\partial}{\partial t}\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH + \nabla\cdot\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH\dot{\mathbf{R}} = \sum_{\mathrm{sp}}\int d\mathcal{W}\, f\frac{\partial H}{\partial t}$$

- evaluate the last term the same way as before
  - note the $\partial\mathcal{L}_f/\partial t$ term survives (note sign!)

- this yields **energy conservation**
  - dependence on $A_\parallel$ gives magnetic energy, on ϕ gives ExB energy

$$\frac{\partial}{\partial t}\left(\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH - \mathcal{L}_f\right) + \nabla\cdot\left(\sum_{\mathrm{sp}}\int d\mathcal{W}\, fH\dot{\mathbf{R}} + \frac{\partial\mathcal{L}}{\partial\nabla\phi}\frac{\partial\phi}{\partial t} + \frac{\partial\mathcal{L}}{\partial\nabla A_\parallel}\frac{\partial A_\parallel}{\partial t}\right) = 0$$

same result using brackets

- multiply gyrokinetic equation by H, using $\{H, H\} = 0$

$$\frac{\partial}{\partial t} fH + \{H, fH\} = f\frac{\partial H}{\partial t}$$

- functional derivative result

$$\sum_{\mathrm{sp}} \int d\mathcal{W}\, f\frac{\partial H}{\partial t} = \nabla \cdot (\ldots) + \frac{\partial \mathcal{L}_f}{\partial t}$$

- put it in, note how the time derivative of $\mathcal{L}_f$ survives

$$\frac{\partial}{\partial t}\left[\sum_{\mathrm{sp}} \int d\mathcal{W}\, fH - \mathcal{L}_f\right] = \nabla \cdot (\ldots)$$

Canonical Toroidal Momentum Transport

- simplified version assumes field dependence like $H(\phi, A_\|, \nabla\phi, \nabla A_\|)$
  - (general: add $\nabla_\perp^2$ dependence, adds to divergence term)

- use the functional derivative result to evaluate the $\partial f H/\partial\varphi$ term

$$\sum_{\mathrm{sp}} \int d\mathcal{W}\, f \frac{\partial H}{\partial \varphi} = \frac{\partial \mathcal{L}_f}{\partial \varphi} - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial \nabla\phi} \frac{\partial \phi}{\partial \varphi} + \frac{\partial \mathcal{L}}{\partial \nabla A_\|} \frac{\partial A_\|}{\partial \varphi} \right)$$

- the flux surface average will annihilate $\partial/\partial\varphi$

- this will leave only the divergence term

- the $\partial f H/\partial\varphi$ term gives ExB Reynolds and Maxwell stresses generally
  - which add to the direct perp/parallel transport terms

- first we will define the flux surface average . . .

Flux Surface Average

- average over all phase space coordinates except volume V
  - species sum is included (and in below forms, understood)

- flux surface average annihilates divergences except $\partial/\partial V$, commutes past $\partial/\partial t$
  - $\int d\mathcal{W}/B_\parallel^*$ commutes past all spatial derivatives, then do the angle derivatives

- flux surface average of kinetic quantities implies the species sum, hence ...

$$\frac{\partial}{\partial t}(f\sqrt{g}B_\parallel^* P_\varphi) + \frac{\partial}{\partial \mathbf{Z}_p}\cdot\left(f\sqrt{g}B_\parallel^* P_\varphi \dot{\mathbf{Z}}_p\right) = -(f\sqrt{g}B_\parallel^*)\frac{\partial H}{\partial \varphi}$$

... becomes ...

$$\frac{\partial}{\partial t}\langle f P_\varphi\rangle + \frac{\partial}{\partial V}\left\langle f P_\varphi \dot{V}\right\rangle + \left\langle f\frac{\partial H}{\partial \varphi}\right\rangle = 0$$

- evaluate the last term, include in the divergence

  this is **canonical toroidal momentum transport**

$$\frac{\partial}{\partial t}\langle f P_\varphi\rangle + \frac{\partial}{\partial V}\left(\left\langle f P_\varphi \dot{V}\right\rangle - \left\langle \frac{\partial \mathcal{L}}{\partial \nabla\phi}\frac{\partial \phi}{\partial \varphi} + \frac{\partial \mathcal{L}}{\partial \nabla A_\parallel}\frac{\partial A_\parallel}{\partial \varphi}\right\rangle\right) = 0$$

Plasma Momentum — Polarisation

- canonical momentum under quasineutrality is just the plasma momentum
- define the polarisation and the gyrocenter charge density ...

$$\nabla\cdot\mathbf{P} = \sum_{\rm sp}\int d\mathcal{W}\,fe \qquad\qquad \mathbf{P} = -\frac{\delta\mathcal{L}}{\delta\nabla\phi}$$

... write the charge conservation equation ...

$$\frac{\partial}{\partial t}\nabla\cdot\mathbf{P} + \nabla\cdot\left(\sum_{\rm sp}\int d\mathcal{W}\,fe\dot{\mathbf{R}}\right) = 0$$

... then take the flux surface average to get its transport equation

$$\frac{\partial}{\partial V}\left\langle\frac{\partial P^V}{\partial t} + fe\dot{V}\right\rangle = 0$$

Plasma Momentum — Radial Current

- the vorticity transport equation is

$$\frac{\partial}{\partial V}\left\langle \frac{\partial P^V}{\partial t} + fe\dot{V} \right\rangle = 0$$

- assume $\langle\,\rangle$ to vanish at $V = 0$, integrate dV, therefore $\langle\,\rangle$ vanishes everywhere

$$\left\langle \frac{\partial P^V}{\partial t} + fe\dot{V} \right\rangle = 0$$

- this is the statement that the flux surface averaged radial current vanishes

we will use this to convert canonical momentum
to plasma momentum in the transport

Plasma Momentum — ExB Component

- content of canonical momentum is the phase space integral, split into pieces

$$\mathcal{P} = \sum_{\mathrm{sp}} \int d\Lambda\, f\, P_\varphi = \sum_{\mathrm{sp}} \int d\Lambda\, f \left(\frac{e}{c} A_\varphi + p_z b_\varphi \right)$$

pull the A_φ flux fuction out and take the flux surface average

$$\mathcal{P} = \int d\mathcal{V} \left(\frac{1}{c} A_\varphi \left\langle f e \right\rangle + \left\langle f p_z b_\varphi \right\rangle \right)$$

use $\nabla \cdot \mathbf{P}$ for fe, do the divergence by parts, and evaluate

$$\mathcal{P} = \int d\mathcal{V} \left(-\frac{1}{c} \left\langle \mathbf{P} \cdot \nabla A_\varphi \right\rangle + \left\langle f p_z b_\varphi \right\rangle \right)$$

- in an MHD model you can show directly the $\mathbf{P} \cdot \nabla A_\varphi$ piece is ExB momentum

$$\rho_M \frac{c}{B^2} \nabla\phi \cdot \nabla A_\varphi = \cdots = \rho_M (\mathbf{v}_E)_\varphi$$

(detail on the ExB momentum)

- use magnetic field representation

$$\mathbf{B} = I\nabla\varphi + \nabla A_\varphi \times \nabla\varphi$$

- evaluate

$$\frac{c}{B^2}\nabla\phi \cdot \nabla A_\varphi = \frac{c}{B^2} R^2 (\nabla\phi \times \nabla\varphi) \cdot (\nabla A_\varphi \times \nabla\varphi)$$

$$= \frac{c}{B^2} R^2 (\nabla\phi \times \nabla\varphi) \cdot \mathbf{B}$$

$$= \frac{c}{B^2} R^2 (\mathbf{B} \times \nabla\phi) \cdot \nabla\varphi$$

$$\frac{c}{B^2}\nabla\phi \cdot \nabla A_\varphi = R^2 \mathbf{v}_E \cdot \nabla\varphi$$

- finally lower the toroidal angle index

$$\frac{c}{B^2}\nabla\phi \cdot \nabla A_\varphi = (\mathbf{v}_E)_\varphi$$

Canonical Momentum — Polarisation Cancellation

- continuity equation for canonical momentum

$$\frac{\partial}{\partial t}\sum_{\rm sp}\int d\mathcal{W}\, fP_\varphi + \nabla\cdot\sum_{\rm sp}\int d\mathcal{W}\, fP_\varphi\dot{\mathbf{R}} = -\sum_{\rm sp}\int d\mathcal{W}\, f\frac{\partial H}{\partial\varphi}$$

take flux surface average (remember species sum)

$$\frac{\partial}{\partial t}\left\langle fP_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle fP_\varphi\dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle$$

look at the terms involving A_φ (which pulls out of the FS average)

$$\frac{\partial}{\partial t}\frac{A_\varphi}{c}\left\langle fe\right\rangle + \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle fe\dot{V}\right\rangle = \frac{\partial}{\partial V}\frac{A_\varphi}{c}\left\langle\frac{\partial P^V}{\partial t} + fe\dot{V}\right\rangle - \frac{\partial}{\partial t}\frac{1}{c}\left\langle\mathbf{P}\cdot\nabla A_\varphi\right\rangle$$

the first term on the RHS vanishes (radial current)
leaving just the plasma momentum

detail

- we have for the part involving $A_\varphi = \psi$

$$\frac{\partial}{\partial t}\left\langle \frac{e}{c}\psi\, f \right\rangle + \frac{\partial}{\partial V}\left\langle \frac{e}{c}\psi\, f\dot{V} \right\rangle + \cdots$$

- move the e onto the f and put the polarisation divergence into the first term

$$\frac{\partial}{\partial t}\left\langle \frac{1}{c}\psi\nabla\cdot\mathbf{P} \right\rangle + \frac{\partial}{\partial V}\left\langle \frac{1}{c}\psi\, ef\dot{V} \right\rangle + \cdots$$

- integrate the divergence on polarisation by parts

$$\frac{\partial}{\partial t}\left\langle -\frac{1}{c}\mathbf{P}\cdot\nabla\psi \right\rangle + \frac{\partial}{\partial t}\left\langle \nabla\cdot\left(\frac{1}{c}\psi\,\mathbf{P}\right) \right\rangle + \frac{\partial}{\partial V}\left\langle \frac{1}{c}\psi\, ef\dot{V} \right\rangle + \cdots$$

- commute $\partial/\partial t$ through on second term

$$\frac{\partial}{\partial t}\left\langle -\frac{1}{c}\mathbf{P}\cdot\nabla\psi \right\rangle + \left\langle \nabla\cdot\left(\frac{1}{c}\psi\,\frac{\partial\mathbf{P}}{\partial t}\right) \right\rangle + \frac{\partial}{\partial V}\left\langle \frac{1}{c}\psi\, ef\dot{V} \right\rangle + \cdots$$

- average of a divergence is a volume derivative of a volume integral

$$\frac{\partial}{\partial t}\left\langle -\frac{1}{c}\mathbf{P}\cdot\nabla\psi\right\rangle + \frac{\partial}{\partial V}\left\langle \frac{1}{c}\psi\,\frac{\partial P^V}{\partial t}\right\rangle + \frac{\partial}{\partial V}\left\langle \frac{1}{c}\psi\,ef\dot{V}\right\rangle + \cdots$$

- the flux variable ψ pulls out of the average

$$\frac{\partial}{\partial t}\left\langle -\frac{1}{c}\mathbf{P}\cdot\nabla\psi\right\rangle + \frac{\partial}{\partial V}\frac{1}{c}\psi\left\langle \frac{\partial P^V}{\partial t}\right\rangle + \frac{\partial}{\partial V}\frac{1}{c}\psi\left\langle ef\dot{V}\right\rangle + \cdots$$

- the second and third terms combine ...

$$\frac{\partial}{\partial t}\left\langle -\frac{1}{c}\mathbf{P}\cdot\nabla\psi\right\rangle + \frac{\partial}{\partial V}\frac{1}{c}\psi\left\langle \frac{\partial P^V}{\partial t} + ef\dot{V}\right\rangle + \cdots$$

... and vanish

$$\frac{\partial}{\partial t}\left\langle -\frac{1}{c}\mathbf{P}\cdot\nabla\psi\right\rangle + \cdots$$

- this completes the polarisation cancellation

Toroidal Momentum Transport Equation

- we now have the terms with the plasma momentum and transport

$$\frac{\partial}{\partial t}\left\langle f p_z b_\varphi - \mathbf{P}\cdot\nabla A_\varphi \right\rangle + \frac{\partial}{\partial V}\left\langle f p_z b_\varphi \dot{V}\right\rangle = -\left\langle f\frac{\partial H}{\partial \varphi}\right\rangle$$

- put in the functional derivative evaluations for the right side

  this is **toroidal momentum transport**

$$\frac{\partial}{\partial t}\left\langle f p_z b_\varphi - \mathbf{P}\cdot\nabla A_\varphi \right\rangle + \frac{\partial}{\partial V}\left\langle f p_z b_\varphi \dot{V} - \frac{\partial \mathcal{L}}{\partial \nabla\phi}\frac{\partial \phi}{\partial \varphi} - \frac{\partial \mathcal{L}}{\partial \nabla A_\parallel}\frac{\partial A_\parallel}{\partial \varphi}\right\rangle = 0$$

- the transport terms are ExB and magnetic flutter transport of parallel momentum
  - plus the field terms which give ExB Reynolds and Maxwell stresses

there are no spurious terms due to canonical pieces ($A_\varphi = \psi$)

Summary of Momentum Conservation/Transport Results

- the theory has a rigorous basis in Hamilton's Principle and symmetry
- time/toroidal-angle symmetry give energy/toroidal-momentum conservation
- polarisation cancellation yields plasma momentum from canonical momentum
  - Poynting momentum gets dropped by quasineutrality assumptions (small $E^2/8\pi$)
- a well formed plasma toroidal momentum transport equation results
  - no spurious terms due to A_φ (part of canonical momentum)
  - hence no promotion of small, higher-order field terms

polarisation/charge conservation is part of momentum conservation:
if field equation is not consistent with the gyrocenter Lagrangian,
momentum conservation will not be maintained

The Laplacian Term and Reynolds Stress

- allow field dependences $(A, \nabla A, \nabla_\perp^2 A)$ for H and $\mathcal{L}_f$, and define

$$C = -\frac{\partial \mathcal{L}}{\partial A} \qquad N = \frac{\partial \mathcal{L}}{\partial \left|\nabla_\perp A\right|^2/2} \qquad P = -\frac{\partial \mathcal{L}}{\partial \nabla_\perp^2 A}$$

- result we had before, rewritten in terms of C, N, and P

$$\sum_{\text{sp}} \int d\mathcal{W}\, f \frac{\partial H}{\partial \varphi} = \frac{\partial}{\partial \varphi} \left(\mathcal{L}_f - \nabla \cdot P \nabla_\perp A\right) + \nabla \cdot \left[-N \frac{\partial A}{\partial \varphi} \nabla_\perp A - \frac{\partial P}{\partial \varphi} \nabla_\perp A - \frac{\partial A}{\partial \varphi} \nabla_\perp P\right]$$

- take flux surface average

$$\left\langle f \frac{\partial H}{\partial \varphi} \right\rangle = \frac{\partial}{\partial V} \left\langle -\nabla V \cdot \left[N \frac{\partial A}{\partial \varphi} \nabla_\perp A + \frac{\partial P}{\partial \varphi} \nabla_\perp A + \frac{\partial A}{\partial \varphi} \nabla_\perp P\right]\right\rangle$$

The Laplacian Term and Reynolds Stress (2)

- write it again

$$\left\langle f\frac{\partial H}{\partial\varphi}\right\rangle = \frac{\partial}{\partial V}\left\langle -\nabla V\cdot\left[N\frac{\partial A}{\partial\varphi}\nabla_\perp A + \frac{\partial P}{\partial\varphi}\nabla_\perp A + \frac{\partial A}{\partial\varphi}\nabla_\perp P\right]\right\rangle$$

- now together with the rest of the momentum transport equation

$$\frac{\partial}{\partial t}\left\langle f p_z b_\varphi - \mathbf{P}\cdot\nabla A_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle\nabla V\cdot\left[f p_z b_\varphi \dot{\mathbf{R}}\right]\right\rangle$$

$$+ \frac{\partial}{\partial V}\left\langle -\nabla V\cdot\left[N\frac{\partial A}{\partial\varphi}\nabla_\perp A + \frac{\partial P}{\partial\varphi}\nabla_\perp A + \frac{\partial A}{\partial\varphi}\nabla_\perp P\right]\right\rangle = 0$$

- the field fluxes on the second line are all summed over the field variables A
  - usually, the A are ϕ and $A_\parallel$
  - these represent wave-wave nonlinearities

MHD version: Reynolds and Maxwell Stresses

- the toroidal momentum transport equation without $\nabla_\perp^2$ dependences is

$$\frac{\partial}{\partial t}\left\langle f p_z b_\varphi - \mathbf{P}\cdot\nabla A_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle f p_z b_\varphi \dot{V} - \frac{\partial\mathcal{L}}{\partial\nabla\phi}\frac{\partial\phi}{\partial\varphi} - \frac{\partial\mathcal{L}}{\partial\nabla A_\|}\frac{\partial A_\|}{\partial\varphi}\right\rangle = 0$$

- using an "MHD Lagrangian" in a mean-field model (paper) this is evaluated as

$$\frac{\partial}{\partial t}\left\langle\rho_M u_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle\left(\rho_M \widetilde{v}_E^V \widetilde{u}_\| + \widetilde{p}\widetilde{b}^V\right) b_\varphi\right\rangle + \frac{\partial}{\partial V}\left\langle\rho_M \widetilde{v}_E^V \widetilde{v}_{E\varphi}\right\rangle + \frac{\partial}{\partial V}\left\langle\rho_M v_A^2 \widetilde{b}^V \widetilde{b}_\varphi\right\rangle = 0$$

- Reynolds/Maxwell stresses (3rd/4th terms): same sign as in vorticity transport eqn
  - if these drive the flow, they represent negative diffusivity
  - wave fluxes only transport momentum if they take energy out of the flow

- hence one expects the parallel/radial component to be most relevant
  - this component functions if turbulence spatial symmetry is broken
  - symmetry breaking effectively provided by poloidal ExB advection in 2D equilibrium

Electrostatic Model with FLR and ExB-Mach Effects

- FLR terms with ExB Mach number correction (Miyato et al JPSJ 2009)

$$H = e\phi + \frac{\rho_L^2 + \rho_E^2}{4}\nabla_\perp^2(e\phi) - m\frac{u_E^2}{2} \qquad \mathcal{L}_f = 0$$

with particle and ExB gyroadii, ExB kinetic energy and vorticity

$$\rho_L^2 = \frac{2\mu B}{m\Omega^2} \qquad \rho_E^2 = \frac{u_E^2}{\Omega^2} \qquad u_E^2 = \frac{c^2}{B^2}\left|\nabla_\perp\phi\right|^2 \qquad \Omega_E = \frac{c}{B}\nabla_\perp^2\phi$$

- appearance of the ExB Mach (u_E^2) and vorticity (Ω_E) corrections

$$C_E = \sum_{\rm sp} ne \qquad N_E \equiv \sum_{\rm sp}\frac{nmc^2}{B^2}\left(1 - \frac{\Omega_E}{2\Omega}\right) \qquad P_E \equiv \sum_{\rm sp}\frac{mc^2}{2eB^2}\left(p_\perp + \frac{1}{2}nmu_E^2\right)$$

- polarisation equation is $\qquad C_E + \nabla_\perp^2 P_E + \nabla\cdot N_E\nabla_\perp\phi = 0$

- both pieces (ExB and diamagnetic) of Reynolds stress

$$\Pi_E^V = \left\langle -\nabla V\cdot\left[N_E\frac{\partial\phi}{\partial\varphi}\nabla_\perp\phi + \frac{\partial P_E}{\partial\varphi}\nabla_\perp\phi + \frac{\partial\phi}{\partial\varphi}\nabla_\perp P_E\right]\right\rangle$$

FLR polarisation correction

- use the Padé form of arbitrary-$k_\perp$ polarisation

$$H = \cdots - \frac{e^2}{2B}\frac{\partial}{\partial\mu}\left[\phi^2 - (J_0\phi)^2\right]$$

$$H = \cdots - m\frac{c^2}{B^2}\left|\nabla_\perp\phi\right|^2 - \frac{\mu B}{2\Omega^2}\frac{c^2}{B^2}\left|\nabla_\perp^2\phi\right|^2$$

- this merely adds another Ω_E/Ω correction

$$P_E = \sum_{\mathrm{sp}}\frac{mc^2}{2eB^2}\left[p_\perp\left(1 - \frac{\Omega_E}{2\Omega}\right) + nm\frac{u_E^2}{2}\right]$$

- the effects of this are one more order down from the u_E^2 term
  (or for $u_E \sim v_D$, both terms are similarly small compared to $p_\perp$)

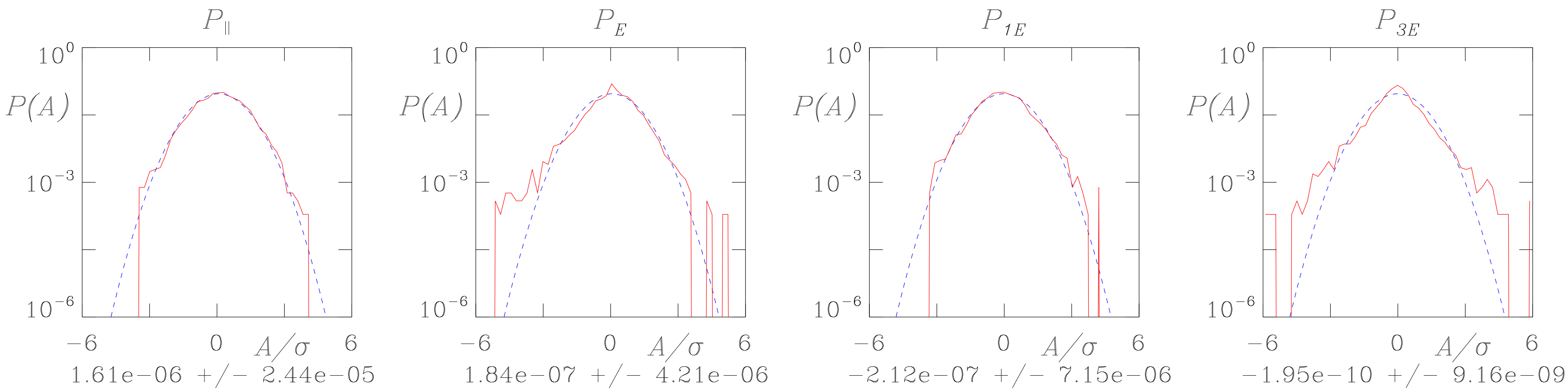
- this and all other corrections are smiliarly down in the
  fundamental small parameter of gyrokinetics: Ω_E/Ω

symmetry results in computations

- important to note: all field transport terms have the same symmetry structure

$$\Pi_E^V = \left\langle -\nabla V \cdot \left[N_E \frac{\partial \phi}{\partial \varphi} \nabla_\perp \phi + \frac{\partial P_E}{\partial \varphi} \nabla_\perp \phi + \frac{\partial \phi}{\partial \varphi} \nabla_\perp P_E \right] \right\rangle$$

- flux distributions for main perp/parallel term, ExB and FLR Reynolds stresses, and ExB Mach correction (delta-f gyrokinetic fluxtube model dFEFI, edge)



- note symmetry means a symmetric PDF, not something spatial

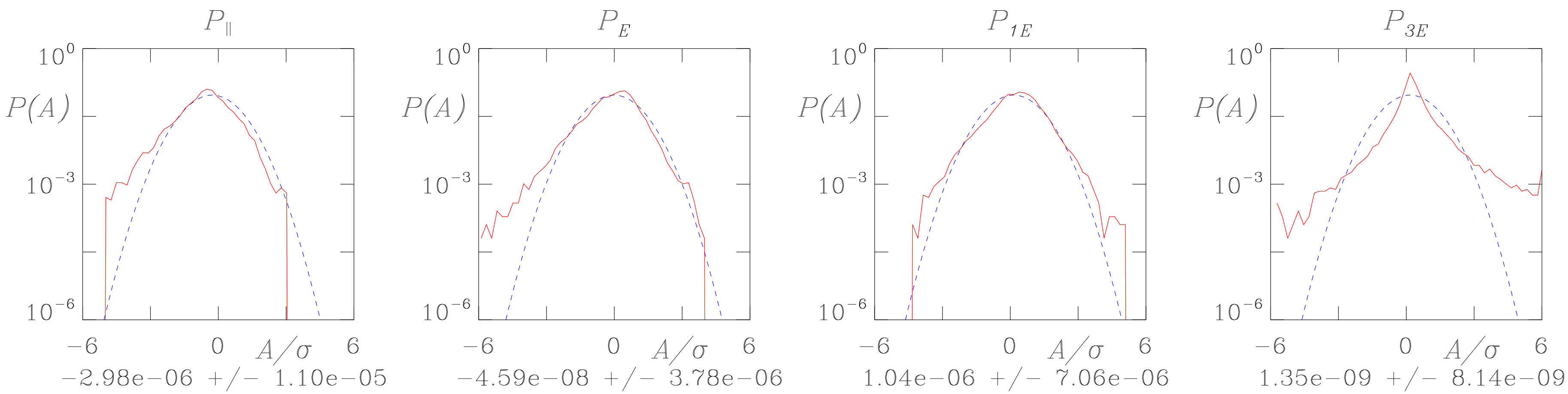
symmetry results in global computations

- from the GEMR global delta-f gyrofluid model (*Phys Plasmas* 17 2010, p102306)

- 2D quasi-equilibrium asymmetry:
  ◦ restore the $\partial/\partial s$ terms neglected in fluxtube models
  ◦ conserve energy (necessary step before continuing)
  ◦ up/down in/out sideband symmetry broken by $\partial/\partial s$ on $m = \pm 1$ sidebands

- Cyclone-like EM case (similar parameters, better q-profile)

- flux distributions for main perp/parallel term, ExB and FLR Reynolds stresses, and ExB Mach correction (GEMR, core)



symmetry results in edge case with self-cons gradients

- from the GEMR global delta-f gyrofluid model (*Phys Plasmas* 17 2010, p102306)

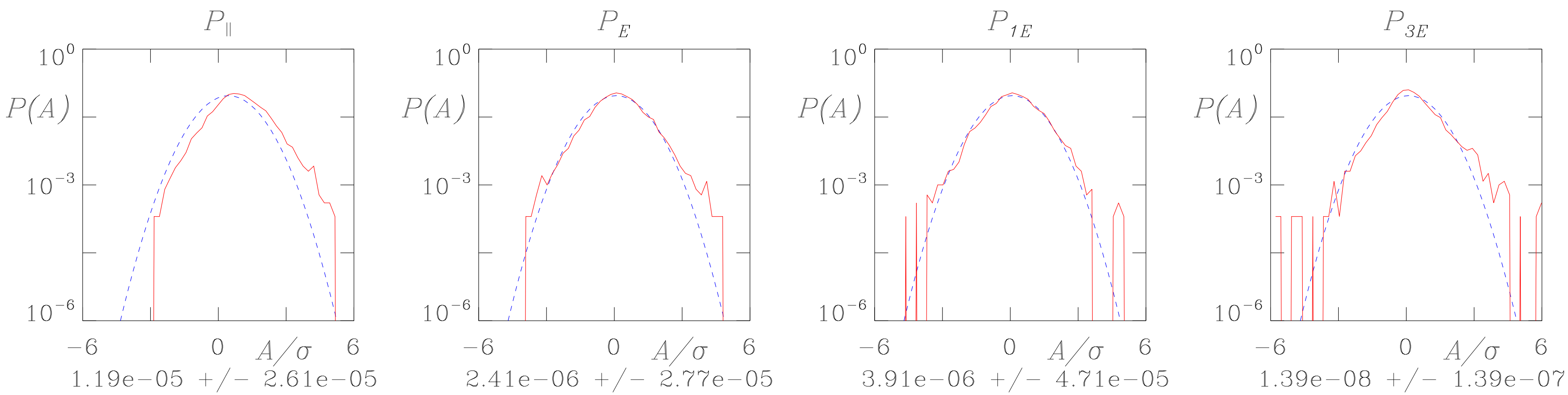
- edge L-mode base case (B Scott, Contrib Plasma Phys 46:714 2006)

$$T_e = T_i = 100\,\mathrm{eV} \qquad n_e = n_i = 2.0 \times 10^{13}\,\mathrm{cm}^{-3} \qquad B = 2.5\,\mathrm{T}$$

$$R = 165\,\mathrm{cm} \qquad L_T = L_\perp = 3.5\,\mathrm{cm} \qquad L_n = 7.0\,\mathrm{cm} \qquad q = 3.5 \qquad \hat{s} = 1.14$$

- normalised parameters $\hat{\beta} = 1.75, \quad \hat{\mu} = 7.41, \quad C = 3.11, \quad \nu_B = 0.13$
  (B Scott Phys Plasmas 6/2005, PPCF 12/2003 and 12/2006)

- flux distributions for main perp/parallel term, ExB and FLR Reynolds stresses, and ExB Mach correction (GEMR, edge)



Main Points Overall

- gyrokinetic theory has firm classical field-theory basis
- toroidal momentum is well conserved globally and locally (plasma as well as canonical)
- momentum and charge conservation (Pol Current) must be considered together
- a good system has equations traceable to a Lagrangian in canonical form
- all terms in the momentum flux have similar symmetry properties
- lowest order terms plus FLR corrections entirely adequate for
  - polarisation
  - momentum transport

a clear demonstration of energetic consistency is
a necessary part of any well posed model

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